

On the symmetries of special holonomy sigma models

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ABSTRACT: In addition to superconformal symmetry, $(1,1)$ supersymmetric two-dimensional sigma models on special holonomy manifolds have extra symmetries that are in one-to-one correspondence with the covariantly constant forms on these manifolds. The superconformal algebras extended by these symmetries close as W-algebras, i.e. they have field-dependent structure functions. It is shown that it is not possible to write down cohomological equations for potential quantum anomalies when the structure functions are field-dependent. In order to do this it is necessary to linearise the algebras by treating composite currents as generators of additional symmetries. It is shown that all cases can be linearised in a finite number of steps, except for G_2 and $SU(3)$. Additional problems in the quantisation procedure are briefly discussed.

KEYWORDS: BRST Symmetry, Sigma Models, Conformal and W Symmetry, Anomalies in Field and String Theories.

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1. Introduction

There has been a long history of interplay between differential geometry and supersymmetric non-linear sigma models starting with the observation that $N = 2$ supersymmetry in two dimensions requires the sigma model target space to be a Kähler manifold [1]. It was first pointed out in [2] that one could construct conserved currents in $(1, 1)$ sigma models given a covariantly constant form on the target space, and in [3] it was shown that the $(1, 1)$ model on a Calabi-Yau three-fold has an extended superconformal algebra involving precisely such a current constructed from the holomorphic three-form. In [4] symmetries of this type were studied systematically in the classical sigma model setting; each manifold on Berger's list of irreducible non-symmetric Riemannian manifolds has one or more covariantly constant forms which give rise to conserved currents and the corresponding Poisson bracket algebras are non-linear, i.e. they are of W-symmetry type. Subsequently the properties of these algebras were studied more abstractly in a conformal field theory framework [5, 6] and more recently in topological models [7].

In this paper we revisit the symmetries of classical $(1, 1)$ supersymmetric non-linear sigma models with target spaces which admit torsion-free connections with special holonomy groups. The structure of the classical Poisson bracket algebra of currents associated with the covariantly constant forms and the supercurrent is investigated and it is shown that, in most cases, it can be linearised by the inclusion of a finite number of composite currents. The exceptional cases are $SU(3)$ and G_2 , possibly the two cases of most interest in string theory. In these cases derivatives of the original currents are generated and the presence of these suggests that finite linearisations may not be possible.

The main motivation for studying these symmetries is as a preparation for trying to gain a better understanding of them at the quantum level. Such an understanding may be of use in the study of higher-order corrections in string theory in the sigma model context [8–10], a topic which has recently received renewed attention from the point of view of spacetime supersymmetry [11].

Since the symmetry transformations associated with covariantly constant forms of degree greater or equal to three are non-linear, even for a flat target space, one might anticipate that BRST techniques would be necessary in their analysis, and since the algebras only close in a field-dependent way one would also expect that the BV version might be helpful. The idea would be to use these techniques in the context of the algebraic renormalisation programme [12] in order to study possible anomalies in a cohomological framework [13]. However, it turns out that this formalism is not sufficient to deal with the problems we are mainly interested in, namely the sigma model either by itself or in the presence of external gauge fields. This motivates the search for linearised extensions of the classical special holonomy W-algebras.

The symmetry transformations associated with covariantly constant forms have the property that their parameters are chiral in the sense that they depend on half of the worldsheet superspace coordinates (see (2.4)). The BRST transformation of the matter field X^i is,¹

$$sX^i = c^A R_A^i(X) \tag{1.1}$$

where c^A are the parameter ghosts. In order that the BRST operator s is nilpotent on X^i we would like the transformation of the ghosts to take the standard form

$$sc^A = \frac{1}{2} c^C c^B f_{BC}^A(X), \tag{1.2}$$

where $f_{BC}^A(X)$ are the field-dependent structure functions defined by $[R_B, R_C] = f_{BC}^A R_A$, the R_A s being regarded as vector fields on the space of sigma model fields. However, this is not consistent because one is transforming chiral objects into non-chiral ones. The structure functions depend on the currents and are only chiral on-shell. This then prevents one from writing down a classical master equation in the chiral, or superconformal, theory.

One can circumvent this problem by gauging the algebra in which case the ghosts are no longer taken to be chiral. In the context of a $d = 2$ superconformally invariant action $S_0(X)$, for a single chiral sector,² this involves the modification of the original action S_0 by

$$S_0(X) \rightarrow S_0(X) + h^A j_A(X), \tag{1.3}$$

where j_A are the conserved currents, and the gauge fields h^A are required to transform as

$$sh^A = D_- c^A + h^C c^B f_{BC}^A(X). \tag{1.4}$$

Two further problems now present themselves. The first is that the algebra of transformations may fail to close on the gauge fields and the ghosts due to relations between the currents which show up in the Jacobi identities. This is discussed in more detail in section 4. Even when this is not a problem the BRST transformations (1.1), (1.2), (1.4) involve the background fields, c^A and h^A , transforming into expressions containing quantum fields. It is possible to construct a solution to the classical master equation, but the naive Ward

¹The deWitt notation is being used temporarily, with repeated indices implying integration as well as summation over labels. See for example [14].

²For both sectors the gauging procedure is more complicated - see [15, 16].

identities involve insertions of composite operators [17]. It is not known how to analyse anomalies using cohomological techniques in this situation and it is therefore difficult, if not impossible, to carry through the algebraic renormalisation programme order by order in perturbation theory. On the other hand, a perturbative evaluation of the explicit non-local expressions involving composite operators is hopelessly difficult in the context of a generic special holonomy sigma model.

As far as we have been able to ascertain there seems to be no way out of this apart from linearisation. For the cases where we can establish finite linearisations at the classical level it should be possible to analyse the renormalisation of the symmetries of the effective action itself in perturbation theory. We discuss this further in the final section. However, we note that in order to analyse anomalies in the current algebra, or in the operator product expansion, sources for the currents need to be introduced which implies that the gauging (1.3) is necessary. So even though writing down OPE expressions involving composite currents is commonplace in the abstract CFT, for special holonomy sigma models it seems that the OPE can really only be understood explicitly in perturbative renormalisation theory if the algebra can be linearised. Moreover, as we shall discuss later, some of the finite linearisations turn out to be unstable in the quantum theory due to the fact that operators which do not appear in the classical Poisson bracket algebra can be generated by the OPE.

2. Review of basics

The action for a (1,1)-supersymmetric sigma model without boundary is

$$S = \int dz g_{ij} D_+ X^i D_- X^j, \tag{2.1}$$

where g_{ij} is a Riemannian metric on the n -dimensional target space M . $X^i, i = 1, \dots, n$, is the sigma model field represented in some local chart for M and z denotes the coordinates of (1,1) superspace Σ . We shall use a light-cone basis so that $z = (x^{++}, x^{--}, \theta^+, \theta^-)$, with $x^{++} = x^0 + x^1, x^{--} = x^0 - x^1$. D_+ and D_- are the usual flat superspace covariant derivatives which obey the relations

$$D_+^2 = i\partial_{++}; \quad D_-^2 = i\partial_{--}; \quad \{D_+, D_-\} = 0. \tag{2.2}$$

We use the convention that $\partial_{++} x^{++} = 1$. We shall take the superspace measure to be

$$dz := d^2x D_+ D_- \tag{2.3}$$

with the understanding that the superfield obtained after integrating over the odd variables (i.e after applying $D_+ D_-$ to the integrand) is to be evaluated at $\theta = 0$.

The action (2.1) is invariant under superconformal transformations which act independently on the left (+) and right (-) light-cone sectors. In the left sector, a superconformal transformation takes the form

$$\delta X^i = 2a_{--} \partial_{++} X^i - iD_+ a_{--} D_+ X^i, \tag{2.4}$$

where the parameter a_{--} is chiral, $D_- a_{--} = 0$. The corresponding supercurrent is the energy-momentum tensor

$$T_{+3} := g_{ij} \partial_{++} X^i D_+ X^j, \tag{2.5}$$

The current is conserved in the sense that $D_- T_{+3} = 0$ on-shell. Similarly, there is a conserved energy-momentum tensor T_{-3} in the right sector.

We shall say that the target space has special holonomy if there are one or more covariantly constant forms which reduce the corresponding holonomy groups from $SO(n)$ to groups G on Berger's list. These are: $U(m)$ and $SU(m)$ for $n = 2m$; $Sp(k)$ and $Sp(k) \cdot Sp(1)$ for $n = 4k$; G_2 and $Spin(7)$.

Let L be a vector-valued l -form such that the $l+1$ -form obtained by lowering the vector index (taken to be in the first slot) is covariantly constant; this form will also be denoted L . (It should be clear from the context which is meant). The symmetry transformation associated with L is

$$\delta_L X^i = a_L L^i{}_L D_+ X^L \tag{2.6}$$

where the parameter a_L has Lorentz weight $-l$ and Grassmann parity $(-1)^l$ and the multi-index L denotes l antisymmetrised indices, $L := [l_1 \dots l_l]$. We shall use the notation L_2 to denote antisymmetrisation over the $l-1$ indices beginning with l_2 , and so on. Under a general variation of the field X the change in the action is

$$\begin{aligned} \delta S &= \int dz 2g_{ij} \delta X^i \nabla_- D_+ X^j \\ &= - \int dz 2g_{ij} \delta X^i g_{ij} \nabla_+ D_- X^j. \end{aligned} \tag{2.7}$$

If we substitute (2.6) into the top line of (2.7) we see that $\delta S = 0$ provided that the parameter is chiral, $D_- a_L = 0$. The corresponding conserved current will also be denoted by L ; it satisfies $D_- L = 0$ on-shell and is given by

$$L = \frac{1}{l+1} L_{iL} D_+ X^{iL}. \tag{2.8}$$

In order to evaluate the commutator, of two such transformations one needs some algebraic relations which can be proved for any special holonomy forms. If we set

$$(L \cdot M)_{iL_2, jM_2} := L_{kiL_2} M^k{}_{jM_2}, \tag{2.9}$$

then one can verify that

$$\begin{aligned} (L \cdot M)_{i[L_2, jM_2]} &= (-1)^{l+1} P_{ijL_2M_2} + \frac{m}{2} g_{i[j} Q_{L_2M_2]}, \\ (L \cdot M)_{[jL_2, i]M_2} &= (-1)^l P_{ijL_2M_2} + \frac{l}{2} g_{i[j} Q_{L_2M_2]}, \\ (L \cdot M)_{i[L_2, j]M_2} + (i \leftrightarrow j) &= g_{ij} Q_{L_2M_2} - (l+m-2) g_{(i[l_2} Q_{j)L_3M_2)}. \end{aligned} \tag{2.10}$$

The tensors P and Q are totally antisymmetric and covariantly constant; in particular cases they can vanish. Both of them can be used to define L -type symmetry transformations,

but in the commutator of two special holonomy transformations, $[\delta_L, \delta_M]$, Q is combined with the energy-momentum tensor. After some algebra one finds that

$$[\delta_L, \delta_M]X^i = \delta_P X^i + \delta_K X^i, \tag{2.11}$$

where each term is now a symmetry by itself. The P transformation, which is of standard L -type has parameter a_P given by

$$a_P = (-1)^{l+1} m a_M D a_L - (-1)^m l D a_M a_L. \tag{2.12}$$

The K transformation is defined as follows. If we set

$$K_{i,K} := g_{i[k_1} Q_{K_2]}, \tag{2.13}$$

where the multi-index K takes on $l + m - 1$ values, then it is not difficult to show (for any covariantly constant antisymmetric tensor Q) that

$$\delta_K X^i = \frac{l + m - 1}{l + m - 2} \left(a_K K_j^i{}_{K_2} \partial_{++} X^j D_+ X^{K_2} + \frac{i(-1)^k}{k} K^i{}_K \nabla_+ (a_K D_+ X^K) \right) \tag{2.14}$$

is a symmetry of the action (2.1). In fact, the corresponding conserved quantity is the composite current TQ . For the case in hand the parameter a_K is

$$a_K = i(-1)^{l+1} \frac{lm(l + m - 2)}{2} a_M a_L. \tag{2.15}$$

3. Poisson bracket algebras

In this section we re-examine the algebra of symmetry transformations for the torsion-free model. The idea is to try to linearise the W -type algebraic structure by treating any composite currents as new independent generators. We shall see that for $SU(m), m \geq 4$, $Sp(k) \cdot Sp(1)$ and for $Spin(7)$ it turns out to be rather simple to do this by including a small number of extra generators. Since the $Sp(k)$ case is linear anyway ($N = 4$ superconformal symmetry), this only leaves two cases which cannot be linearised straightforwardly, namely $SU(3)$ and G_2 . The problem here is that derivatives of the original currents turn up and this interferes with the finiteness which is otherwise due to the fact that differential forms only have finite degree.

The subject is best studied using Poisson brackets; these were introduced in [4]. These brackets are based on the observation that $(1, 1)$ superspace factorises, $\Sigma = \Sigma^{(+)} \times \Sigma^{(-)}$, so that we can view $z^{(-)} := (x^{--}, \theta^-) \in \Sigma^{(-)}$ as the super-time, while the other coordinates $z^{(+)} := (x^{++}, \theta^+) \in \Sigma^{(+)}$ are spatial coordinates on which the fields depend. On-shell the currents depend only on the latter as they are conserved in super-time. In the following discussion the minus coordinates are irrelevant, so that we can drop the pluses from the formulae without loss of clarity. In this section, therefore, D will denote D_+ while ∂ denotes ∂_{++} , with $D^2 = i\partial$.

The basic Poisson bracket (PB) is

$$(DX^i(1), DX^j(2)) = g^{ij} \nabla_1 \delta_{12} \tag{3.1}$$

where (1, 2) refer to two different points in $\Sigma^{(+)}$, $\nabla = \nabla_+$ and δ_{12} is the delta-function in $\Sigma^{(+)}$ which, as there is only one odd coordinate, is Grassmann odd. As all of the tensors appearing in the currents are covariantly constant, the covariant derivative in the basic PB can be replaced by the ordinary derivative, and the tensors can be regarded as constants. With this being understood one can write (3.1) and its corollaries as

$$\begin{aligned}
 (DX^i(1), DX^j(2)) &= g^{ij} D_1 \delta_{12} \\
 (\partial X^i(1), DX^j(2)) &= g^{ij} \partial_1 \delta_{12} \\
 (DX^i(1), \partial X^j(2)) &= -g^{ij} \partial_1 \delta_{12} \\
 (\partial X^i(1), \partial X^j(2)) &= ig^{ij} \partial_1 D_1 \delta_{12} .
 \end{aligned}
 \tag{3.2}$$

In the following we shall write $j(a)$ to mean a smeared current. For each current the parameter a has the opposite Grassmann parity, so we have

$$j(a) = \int dz j(z)a(z) = \int dz a(z)j(z),
 \tag{3.3}$$

where z now denotes $z^{(+)}$. For any three currents A, B, C and parameters f, g we have

$$\begin{aligned}
 (A(f), BC(g)) &= (A(f), B(Cg)) + (-1)^{BC} (A(f), C(Bg)) \\
 (AB(f), C(g)) &= (B(fA), C(g)) + (-1)^{AB} (A(fB), C(g)) .
 \end{aligned}
 \tag{3.4}$$

three basic PBs are the superconformal algebra,

$$(T(a), T(b)) = T(2(\partial ab - a\partial b) + iDaDb),
 \tag{3.5}$$

the PB of the supercurrent with an L -current L ,

$$(T(a), L(b)) = L(l\partial ab - 2a\partial b + iDaDb),
 \tag{3.6}$$

and the PB of two currents, L, M ,

$$(L(a_L), M(a_M)) = -P(a_P) - TQ(a_K),
 \tag{3.7}$$

where a_P and a_K are defined in (2.12) and (2.15) respectively. The programme now is to compute the PBs of the composite TQ with all of the other currents including itself. It is easier to do this explicitly case by case (recall that $\dim M = n$).

$G = \mathbf{U}(m); n = 2m$. When $G = \mathbf{U}(m)$ there is one extra current associated with the complex structure J and the algebra is just the $N = 2$ superconformal algebra. In the present notation this is, in addition to the $N = 1$ PB (T, T) ,

$$\begin{aligned}
 (T(a), J(b)) &= J(\partial ab - 2a\partial b + iDaDb), \\
 (J(a), J(b)) &= -iT(ab),
 \end{aligned}
 \tag{3.8}$$

where the current is $J = \frac{1}{2}J_{ij}DX^{ij}$. The pair (J, T) together make up the $N = 2$ supercurrent which can be viewed as a real $N = 2$ superfield. There are further $N = 2$ multiplets given by pairs of the form (J^p, TJ^{p-1}) . The PB algebra generated by these currents closes,

$$\begin{aligned} (J^p(a), J^q(b)) &= -ipqTJ^{p+q-2}(ab) \\ (TJ^p(a), J^q(b)) &= \frac{J^{p+q}}{p+q}(q(2q-1)\partial ab - 2q(p+1)a\partial b + iqDaDb) \\ (TJ^p(a), TJ^q(b)) &= TJ^{p+q}((2q+2)\partial ab - (2p+2)a\partial b + iDaDb) . \end{aligned} \quad (3.9)$$

$G = \mathbf{Sp}(k); n = 4k$. When $G = \mathbf{Sp}(k)$, so that M is a hyperKähler manifold, we have three complex structures $\{J_r\}$, $r = 1, 2, 3$ giving rise to the $N = 4$ superconformal algebra with

$$(J_r(a), J_s(b)) = -i\delta_{rs}T(ab) + \epsilon_{rst}J_t(Dab + aDb) . \quad (3.10)$$

$G = \mathbf{Sp}(k) \cdot \mathbf{Sp}(1); n = 4k$. The holonomy groups $\mathbf{Sp}(k) \cdot \mathbf{Sp}(1)$, which correspond to quaternionic Kähler geometries in $4k$ dimensions, give rise to W-type algebras which admit finite linearisations. There is a set of three complex structures $\{J_r\}$ but they are not globally defined on the target space. This means that one cannot define three additional supercurrents. However, there is a covariantly constant four-form $\omega_L = \omega_r \wedge \omega_r$, where ω_r is the local two-form corresponding to J_r . This gives rise to an L -type symmetry and hence we have an $N = 1$ superconformal algebra extended by this current. The full set of currents is given by $\{L^p, TL^q; p = 1, \dots, k; q = 1 \dots k - 1\}$.

The PB of two L -currents is

$$(L(a), L(b)) = -4iTL(ab) . \quad (3.11)$$

Using this result and the PB of T with L one can verify straightforwardly that

$$\begin{aligned} (L^p(a), L^q(b)) &= -4ipqTL^{p+q-1}(ab) \\ (TL^p(a), L^q(b)) &= \frac{L^{p+q}}{p+q}(q(4q-1)\partial ab - 2q(2p+1)a\partial b + iqDaDb) \\ (TL(a), TL(b)) &= TL^{p+q}((4q+2)\partial ab - (4p+2)a\partial b + iDaDb) . \end{aligned} \quad (3.12)$$

$G = \mathbf{SU}(m); n = 2m$. When the holonomy group is $\mathbf{SU}(m)$, $m \geq 3$ the target space is a Calabi-Yau manifold. As well as a complex structure there is a covariantly constant $(m, 0)$ -form Ω . We shall work with L and \hat{L} which are respectively the real and imaginary parts of Ω . So the generating set of currents is $\{T, J, L, \hat{L}\}$, and we shall set $m = l + 1$ to be in line with our previous conventions. T and J generate an $N = 2$ superconformal algebra and the pair $\{L, \hat{L}\}$ transform as a chiral $N = 2$ superconformal field,

$$\begin{aligned} (J(a), L(b)) &= \hat{L}(lDab + aDb) \\ (J(a), \hat{L}(b)) &= -L(lDab + aDb) . \end{aligned} \quad (3.13)$$

For m even the PBs for the L s are

$$\begin{aligned} (L(a), L(b)) &= -ill!TJ^{l-1}(ab) \\ (\hat{L}(a), \hat{L}(b)) &= -ill!TJ^{l-1}(ab) \\ (\hat{L}(a), L(b)) &= !lJ^{l-1}(Dab + aDb) , \end{aligned} \quad (3.14)$$

while for m odd they are

$$\begin{aligned}
 (L(a), L(b)) &= !!J^{l-1}(Dab - aDb) \\
 (\widehat{L}(a), \widehat{L}(b)) &= !!J^{l-1}(Dab - aDb) \\
 (\widehat{L}(a), L(b)) &= -ill!TJ^{l-1}(ab) .
 \end{aligned}
 \tag{3.15}$$

In both cases the new currents at this level are $K := TJ^{m-2}$ and $M := J^{m-1}$. As noted above this pair forms an $N = 2$ supermultiplet of lowest spin l . From (3.9) we can see that these currents have vanishing PBs with themselves unless $m = 3$ in which case $(TJ, J^2) \sim J^3$. Moreover, the commutators of powers of J and their products with T with L and \widehat{L} are mostly zero. One has

$$(J^p(a), L(b)) = -2i\delta_{p2}TL(ab), \tag{3.16}$$

which comes about using the fact that

$$JL = 0, \tag{3.17}$$

as ω is a (1,1) form while ω_L is the sum of $(m, 0)$ and $(0, m)$ parts, and

$$(TJ^p(a), L(b)) = \delta_{p1}(\partial JL + DT\widehat{L})(lab) - T\widehat{L}(aDb). \tag{3.18}$$

In order to show this one has to use the identity

$$DJL = iT\widehat{L}. \tag{3.19}$$

We therefore see that only TJ and J^2 have non-trivial PBs with L and \widehat{L} , and J^2 only produces J^3 . At the classical level, this pair is only generated for $m = 3$ and so we conclude that $SU(m)$ holonomy algebras have finite linearisations for $m \geq 4$. For $m = 3$, however, there is a new operator which involves a derivative.

Spin(7). The *Spin(7)* case is similar to $Sp(k) \cdot Sp(1)$. This is an $N = 1$ superconformal algebra extended by a superfield current L of weight 2. The invariant form is the self-dual four-form Φ . The Poisson bracket of L with itself gives rise to the composite current $K = TL$, and the PB of this with L gives L^2 which is simply the current associated with the volume form. The new composites are superconformal fields and have vanishing PBs with L and each other. Explicitly we have

$$\begin{aligned}
 (T(a), L(b)) &= L(3\partial ab - 2a\partial b + iDaDb) \\
 (L(a), L(b)) &= 9iTL(Dab) \\
 (L(a), TL(b)) &= L^2(3\partial ab - \frac{3}{2}a\partial b - \frac{i}{2}DaDb) \\
 (L(a), L^2(b)) &= 0 \\
 (TL(a), L^2(b)) &= 0 .
 \end{aligned}
 \tag{3.20}$$

This result depends only on the algebraic relations $T^2 = L^3 = TL^2 = 0$. In fact, this algebra differs from the $Sp(2) \cdot Sp(1)$ algebra only in the coefficients.

G_2 . The other exceptional special holonomy group is G_2 in seven dimensions. This has an invariant three-form φ and its dual is an invariant four-form; they can be combined to give the $Spin(7)$ four-form Φ . However, in this case the PB algebra generated by T , L (corresponding to φ) and M (corresponding to $*\varphi$) leads to derivatives of the original currents and we are unable to conclude that there is a finite linearisation. In more detail, the basic PBs of the weight $3/2$ and 2 currents L and M are

$$\begin{aligned} (L(a), L(b)) &= 2M(Dab - aDb) \\ (L(a), M(b)) &= -18iTL(ab) \\ (M(a), M(b)) &= -24iTM(ab) . \end{aligned} \tag{3.21}$$

The PBs of L, M with TL, TM then give

$$\begin{aligned} (L(a), TL(b)) &= TM\left(\frac{2}{3}Dab - 2aDb\right) + 2(\partial LL + DTM)(ab) \\ (L(a), TM(b)) &= \frac{3}{7}LM(6\partial ab - 2a\partial b + iDaDb) \\ (M(a), TL(b)) &= 0 \\ (M(a), TM(b)) &= 0 . \end{aligned} \tag{3.22}$$

To show these it is necessary to make use of the obvious algebraic identities, such as $L^2 = 0$, as well as the less obvious ones

$$\begin{aligned} DLL &= \frac{4i}{3}TM \\ LDM &= -\frac{4}{7}D(LM); & DLM &= \frac{3}{7}D(LM) \\ L\partial M &= \frac{4}{7}\partial(LM); & \partial LM &= \frac{3}{7}\partial(LM) \end{aligned} \tag{3.23}$$

The PBs of the bilinears are

$$\begin{aligned} (TL(a), TL(b)) &= -T(7\partial LL + \frac{8}{3}DTM)(ab) \\ (TL, TM) &= (TM, TM) = (TL, LM) = (TM, LM) = (LM, LM) = 0 . \end{aligned} \tag{3.24}$$

After a little algebra one can show that the derivative operator in (L, TL) can be replaced by A which is defined to be the primary part of $DLL + \frac{2}{3}DTM$, i.e. it transforms as a primary of weight $\frac{7}{2}$ under superconformal transformations and that the right-hand side of (TL, TL) is proportional to TA . To this level, we therefore find that the only non-algebraic operator that occurs is A together with TA . Unfortunately, the algebra does not close on this set and higher derivative operators are generated. It seems highly unlikely that there is a finite linearisation in this case, or for CY3 which is similar in structure.

4. Jacobi identities for W-algebras

In the BRST/BV language the Jacobi identities (JIs) for the commutator are written as

$$Z^A R_A^i := c^D c^C c^B (f_{BC}^E f_{ED}^A - R_B^j \partial_j f_{CD}^A) R_A^i = 0 . \tag{4.1}$$

If Z^A does not vanish, the JIs imply relations between the generators. This can also be seen in the JIs for the Poisson bracket. Given a particular field-dependent algebra one can think of these as abstract relations which hold independently of a particular representation.

In this section we investigate the reducibility relations that occur for special holonomy W-algebras. We will show that when Z^A is not zero for the gauged chiral algebras (see (1.3)), it is not possible to solve the classical master equation without introducing further ghosts.

As noted in the introduction, linearisation is necessary to analyse current algebras in an interacting CFT. For linearised algebras Z^A vanishes, but this new obstruction to be discussed here may be of interest in special circumstances when the composite operator expressions can be evaluated more easily, or in the context of W-strings, when all the currents are imposed as constraints and the h^A s are treated as quantum fields.

The part of the master equation linear in the ghost antifields, c_A^* , contains the term $c_A^* Z^A$, so if Z^A does not vanish there is a potential obstruction to solving it. It turns out that for the gauged chiral systems Z^A is a function of the currents, and terms proportional to $h^* c^*$ can be added to the BV action so that this part of the master equation is satisfied. Alternatively, the field dependent closure functions can be set to zero using appropriate terms proportional to $X^* h^*$.

The closure of the transformations acting on the gauge fields (1.4) involves the JIs, and when $Z^A \neq 0$ the algebra closes only modulo certain symmetries which act only on the gauge fields and which reflect the relations between the currents [18, 19]. These will be called null symmetries. For example, for the SU(3) case the null (BRST) symmetry, (c^0 is a parameter ghost),

$$sh^J = -D(c^0 L) \quad sh^{\hat{L}} = ic^0 T \quad sX^i = 0, \tag{4.2}$$

reflects the relation (3.19):

$$DJL - iT\hat{L} \equiv 0. \tag{4.3}$$

The gauged chiral action, (1.3), is

$$S_0 + h^L L + h^{\hat{L}} \hat{L} + h^T T + h^J J. \tag{4.4}$$

It is clear that (4.2) is not the unique symmetry implied by (4.3). There are many possibilities, all related by transformations which are graded antisymmetric in the equations of motion of the gauge fields. The null symmetries, modulo transformations of this type, are graded symmetric. Therefore they cannot be absorbed by adding terms quadratic in h^* to the BV action, and must be introduced as extra symmetries.

Null symmetries are present in any gauged theory with fermions (due to relations such as $T^2 = 0$), and normally they should be ignored. In conventional gauge theories the properness condition³ ensures the existence of a propagator. In the context of string theory or W-strings the gauge fields are non-propagating, and the propagator for the matter fields exists even if the solution is not proper. Nevertheless, it makes sense to impose the

³See, for example, section 4.3 in [14] for the definition of properness.

properness condition modulo null symmetries. The reason is that, if the null symmetries are incorporated into the theory, we face the problem that they are infinitely reducible. This fact is easy to demonstrate for null symmetries proportional to the gauge field equations of motion, but it is also true when they are not of this form [19, 20]. Even more seriously, it is not clear which null symmetries should be included and which should be ignored. If we include one, one might suppose that we should include them all, but then the theory becomes difficult to manage.

When $Z^A \neq 0$ closure forces the introduction of generators for a subset of the possible null symmetries. In this case it makes sense to relax the properness condition modulo null symmetries to include this particular subset, but the infinite reducibility still poses a serious obstruction to understanding the theory [19]. When gauge fields are treated as background fields a proper solution is not required, and reducibility ghosts need not be introduced.⁴ In the examples we encounter it is possible to close the algebra after introducing a finite number of null symmetries. However, even for the non-proper solutions terms increasingly non-linear in the antifields need to be added to the extended action to solve the master equation at higher orders, and it is not clear whether a finite number of terms is sufficient.

Null symmetries arise in this manner for many of the special holonomy W-algebras. In the case of $SU(m)$, using the expressions (2.5) and (2.8) for the T and L currents, one finds the basic relations (3.17) and (3.19). These are implied abstractly by the JIs only in the case of $SU(3)$. For $SU(4)$ one obtains many relations which follow from (3.17) and (3.19), but which involve higher powers of currents. For example, the $(L, (L, L))$ Poisson bracket JI implies:

$$\begin{aligned} J\partial JL \equiv 0 \quad , \quad TD_+J\hat{L} \equiv 0 \quad , \quad D_+TJ\hat{L} \equiv 0 \quad , \\ J(T\hat{L} - D_+JL) \equiv 0 \quad . \end{aligned} \tag{4.5}$$

For $SU(m)$, $m \geq 5$, the situation changes. The Poisson bracket JIs now involve high enough powers of currents so that it becomes possible to absorb the null symmetries by terms quadratic in h^* . For example, the $(L, (L, L))$ Poisson bracket JI in $SU(5)$ implies

$$J^3\hat{L} \equiv 0 \quad . \tag{4.6}$$

The sh^J part of the null symmetry vanishes identically, and therefore the sh^L part can be absorbed by adding a term proportional to $h_j^*h_L^*$ to the BV action. In these cases a proper solution (modulo null symmetries) to the master equation can be found.

The G_2 case is like $SU(3)$, in that the fundamental relations (3.23) are implied by the Poisson bracket JIs, so there is a correlation between this point and the problems with linearisation. For the rest of the special holonomy cases the Jacobi identities do not imply any relations between the generators. That is to say, for $Spin(7)$ and $Sp(k) \cdot Sp(1)$ there are no problems with the JIs, and one can define the classical W-string BRST operator. However, due to the problem of background fields transforming into the quantum fields it is

⁴Because the h^A s are non-propagating fields, there is also the option that a non-proper solution would make some sense even when the h^A are treated as quantum fields. This has not been investigated in the literature.

still not possible to calculate potential anomalies in the OPE using standard cohomological methods; one can only do so at the linearised level. For $SU(3)$ and G_2 finite linearisations are not possible and this makes cohomological analysis of anomalies very difficult.

5. Discussion

We have seen that all the classical special holonomy algebras admit finite linearisations, except for G_2 and $SU(3)$. Thus in all cases except for these we can set up and analyse potential anomalies using cohomological methods. The simplest case to consider is the chiral symmetry algebra for flat target space models. The action is

$$S = S_0 + X_i^* s X^i + c_A^* s c^A, \tag{5.1}$$

where the ghosts and their antifields are chiral. The ghost term in the action gets no quantum corrections and is only introduced to tidy up the algebra. Since the theory defined by S_0 is free it is not affected by anomalies, but there could be anomalies in the antifield sector related to those of the current algebra.

The next step is to gauge models of this type. The action is

$$S = S_0 + h^A j_A + X_i^* s X^i + c_A^* s c^A + h_A^* s h^A, \tag{5.2}$$

where the ghosts are now no longer chiral. Again the last two terms do not receive quantum corrections. Differentiation of the quantum action twice with respect to the gauge fields gives the two-point function of the currents. Since the OPE, which is straightforward to compute in the free theory, introduces the missing J^p and TJ^p currents in $SU(m)$, $m \geq 4$, it follows that this model will be anomalous in these cases. However, for $Spin(7)$ and $Sp(k) \cdot Sp(1)$, one would expect that these models would be anomaly-free.

In principle, one could apply the same ideas to the gauged and ungauged interacting models. However, in practice one has to specify a quantisation procedure. The best is the background field method which allows one to keep track of the geometrical nature of the theory [21]. This can be accomplished by introducing a family of fields $\mathbb{X}(s)$ which interpolate between the background field $X = \mathbb{X}(0)$ and the total field $X_t = \mathbb{X}(1)$. The field $\mathbb{X}(s)$ can be taken to satisfy the geodesic equation, and the quantum field Y is taken to be the tangent vector to the geodesic at $s = 0$. The background-quantum split involves a non-linear shift symmetry [22] which can be shown to be non-anomalous [23]. This symmetry controls the field dependence of the counterterms and can be understood geometrically in terms of the first jet bundle of the tangent bundle. Symmetries of the sigma model can give rise to linear transformations of the quantum field if the symmetry variation commutes with differentiation with respect to s . The condition for this to be the case is that the symmetry variation $\delta\mathbb{X}(s)$ should satisfy the equation of geodesic deviation. Unfortunately, this is not the case for any of the special holonomy symmetry transformations. This implies that the quantum field has to transform non-linearly. In principle, therefore, in order to undertake a complete analysis of the anomalies of the interacting special holonomy sigma models in the background field method one should analyse these non-linear symmetries for graphs with

both background and quantum external lines since the latter can contribute as subgraphs in the effective action for background fields.

Although a full analysis would involve the above steps it is nevertheless not unreasonable to look at the potential anomalies of the background field effective action with no external quantum lines. If one makes a special holonomy transformation of the background field accompanied by the appropriate transformation of the quantum field the local action in the path integral is invariant and the change in the quantum field can be absorbed by a field redefinition in the path integral. Therefore the effective action, defined by

$$e^{i\Gamma[X]} = \int DY e^{iS[X_t]}, \tag{5.3}$$

should satisfy a potentially anomalous Ward Identity of the form

$$W(a_L)\Gamma = \Delta(a_L) \cdot \Gamma \tag{5.4}$$

where a_L is the chiral parameter for an L -type symmetry, $\Delta(a_L)$ is the anomaly, and

$$W(a_L) := \int a_L \delta_L X^i \frac{\delta}{\delta X^i}. \tag{5.5}$$

In this situation one has a Wess-Zumino consistency condition of the form

$$W(a_L)\Delta(a_M) - W(a_M)\Delta(a_L) = \Delta((L(a_L), M(a_M))) . \tag{5.6}$$

For this approach to be valid the algebra in question must be of the linearised type. It would be of interest to investigate this consistency condition. One might expect that special holonomy anomalies would be related to the superconformal anomaly. The above approach can be extended to the gauged case where one would again expect there to be problems for the $SU(m)$ case.

In conclusion, we have seen that it is extremely difficult to analyse the anomalies of special holonomy symmetry W-algebras in the BV framework. It is essential to consider the (chirally) gauged models since the ghosts cannot be consistently taken to be chiral themselves. In these models, however, even in the most favourable cases, such as $Spin(7)$ or $Sp(k) \cdot Sp(1)$, where there are no difficulties due to problems with the Jacobi identity, one is still faced with the problem that the background gauge fields and ghosts transform into the quantum fields. It seems that the only way to avoid this problem in the W-framework is to quantise the gauge fields. Even here, in many cases, one is faced with the problem of infinite reducibility.

These problems can all be avoided to some extent if one utilises the fact that most of the classical special holonomy algebras admit finite linearisations. In the background field method one can then analyse possible anomalies in the effective action with no external quantum lines using the naive classical Ward identities and consistency conditions. On the other hand the inclusion of (background) gauge fields can cause problems due to the fact that the OPE of two currents generates operators which are not in the original set. The only models free from this problem are the $Spin(7)$ and $Sp(k) \cdot Sp(1)$ models.

There are only two models, $SU(3)$ and G_2 , which do not admit finite linearisations. Technically this is because of the presence of currents in the algebra which then generate others with more and more derivatives. An interesting observation is that CY algebras admits closed linear subalgebras generated by $\{T, J, \Omega\}$ or $\{T, J, \bar{\Omega}\}$ where

$$L = \Omega + \bar{\Omega} \quad \hat{L} = i(\Omega - \bar{\Omega}) . \quad (5.7)$$

These can be analysed in all cases including CY3. It is the complex currents Ω which are related to the squares of the spectral flow operator so it may be that commuting the spectral flows in two directions is the source of the problem. The analogous subalgebra in the G_2 case is the tri-critical Ising model [5].

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